VCSMS PRIME

Session 0: Introduction
Carl Joshua Quines
September 21, 2016

Overview

PRIME stands for Program for Improving Mathematical Excellence. A session will consist of roughly one to two hours of problem solving on the selected topic. Problem sets compiled from previous PMOs will be given out at the beginning of the session, and selected questions will be discussed. It is expected that most, if not all, of the problems in a given set should be solved within the week. If you do not know how to solve a problem on any problem set given out, do not hesitate to ask for help.

Schedule

There will be ten sessions this year, followed by the PMO Qualifying Stage. Sessions start as soon as class ends, and will be held at the math lab. Take note that the GMATIC on October 19 is for the whole day, so no session will be held but the problem sets will still be given out.

1. Algebra 1, Wednesday, September 21
2. Trigonometry, Friday, September 23
3. Number theory, Wednesday, September 28
4. Combinatorics 1, Friday, September 30
5. Algebra 2, Wednesday, October 5
6. Combinatorics 2, Friday, October 7
7. Geometry 1, Wednesday, October 12
8. Algebra 3, Friday, October 14
9. Geometry 2, Wednesday, October 19

* GMATIC, Wednesday, October 19

10. Metasolving, Friday, October 21

* PMO Qualifying Stage, Saturday, October 22

Notation

The symbol $\mathbb{R}^*$ is used for the non-zero real numbers, while $\mathbb{R}^+$ is used for the positive real numbers. The natural numbers do not include zero.

The notation $\lfloor x \rfloor$ is the floor function, or the greatest integer function, defined as the greatest integer less than or equal to x. We define $\{x\}$ as the fractional function, or $x - \lfloor x \rfloor$, the fractional part of x. Finally, $\lceil x \rceil$ is the ceiling function, or the least integer function, defined as the least integer greater than or equal to x.

Segments, rays and lines will not be marked with an overbar, and can be deduced from context.

Sources

Problem sources are formatted in the following format:

- One or two digit year number
- Stage: Q for Qualifying, A for Area, N for Nationals
- Substage:
  - For qualifying and area, I for test I, II for test II, III for test III
  - For national written, the substage is omitted.
  - For national orals, E for fifteen-second round, A for thirty-second round, D for sixty-second round
- Problem number

For example, 14QII9 would refer to 2014 qualifying stage test II problem 9, 13AI20 would refer to 2013 area stage test I problem 20, 10NA6 would refer to 2010 nationals oral round thirty-second round problem six, while 9N4 would refer to 2009 nationals written round problem 4.

Miscellaneous

- If you see any typos, errors, broken problems, or if you have any comments, suggestions or corrections, do not hesitate to inform me personally or through `cjquines0@gmail.com`.
- Problems are not copied verbatim, sometimes they are copy-edited to reduce space. The problem is, in essence, the same.
- Please don't be put off by the question just because it is from an area or nationals round. Many area and national oral questions are of similar difficulty to the harder qualifying questions.
- It's okay if you're absent for a session, just make sure to pick up the problem set afterward. Do not be absent for session 10, however, your presence is very important for that session.
- Do try to solve all the problems you can. Yes, I know it's hard because there's a lot of schoolwork, but please do try to make an effort – it takes me time to compile problems as well.

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Session 1: Algebra 1
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Domain and range

1. (11AI9) Find the range of 2^{x^2-4x+1} as x ranges over the real numbers.
2. (13QII6) Find the domain and range of $f(x) = \dfrac{6}{5\sqrt{x^2-10x+29}-2}$.
3. (13NA3) Find the area of the domain of $f(x,y) = \sqrt{25-x^2-y^2} - \sqrt{|x|-y}$.
4. (11NA4) Find the domain of $f(x) = \dfrac{1}{\lfloor x^2-x-2 \rfloor}$.
5. (13QIII3, 14QII1) Find the range of $f(x) = \dfrac{2\cdot 3^{-x}-1}{3^{-x}-2}$ and $g(x) = \dfrac{4^{x+1}-3}{4^x+1}$.
6. (15AI12) Suppose that $1-y = \dfrac{9e^x+2}{12e^x+3}$. Find the integer m such that $m < \dfrac{1}{y} < m+1$ for all real x.
7. (16NE3) Let $f(x) = \ln x$. What are the values of x in the domain of $(f \circ f \circ f \circ f \circ f)(x)$?
8. (13QIII5) Find the range of the following function, where a, b, c are distinct real numbers.
$$f(x) = \frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)}.$$

Logarithms

1. (13QII1) Find $\log_2\left(2^3 \cdot 4^4 \cdot 8^5 \cdot \cdots \cdot (2^{20})^{22}\right)$.
2. (9QI7) How many real roots does $\log_{(x^2-3x)^3} 4 = \dfrac{2}{3}$ have?
3. (10NE14) How many times does the graph of $y = |\log_{\frac{1}{2}}|x|| - 1$ cross the x-axis?
4. (15AI13) The product of two roots of $\sqrt{2014}x^{\log_{2014} x} = x^{2014}$ is an integer. Find its units digit.
5. (13QI3) Given $xy = 10^a, yz = 10^b, zx = 10^c$, find $\log x + \log y + \log z$.
6. (10NE10) Given $a = \log_{14} 16$, express $\log_8 14$ in terms of a.

Exponents

1. Solve for x:
   a) (11QI3) $2^{2^x} = 4^3$
   b) (9QI14) $x^x = x^2$
   c) (9AI6) $x^{x^x} = (x^x)^x$
   d) (10NE3) $x^{x^{2010}} = x^{2010}$
2. (10NE1) Find the smallest integer n such that $n^{300} > 3^{500}$.
3. (16QII7) Arrange from least to greatest: $25^{12}, 16^{14}, 11^{16}$.
4. (11QII9) Given $9^{2x} - 9^{2x-1} = 8\sqrt{3}$, find $(2x-1)^{2x}$.

More logarithms

1. (15AII1) Arrange in ascending order: $\log_3 2, \log_5 3, \log_{625} 75, \dfrac{2}{3}$.
2. (13QI12) Given $\dfrac{\log_2 x}{\log_2 2x - \log_8 2} = 3$, find $1 + x + x^2 + \cdots$.
3. (9NA6) Suppose that $a \geq b > 1$. Find the maximum value of $\log_a \dfrac{a}{b} + \log_b \dfrac{b}{a}$.
4. (11NE6) There exists positive integers k, m, n whose greatest common divisor is 1 such that $k \log_{400} 5 + m \log_{400} 2 = n$. Find $k + m + n$.
5. (11NE12) Let $f(m) = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$, where there are m twos. Find the least integer m such that $\log f(m) > 6$.
6. (14AI9) Solve for $x : \log(5^{\frac{1}{x}} + 5^3) < \log 6 + \log 5^{\left(1 + \frac{1}{2x}\right)}$.
7. (11QI15) Solve for $x : \log x \geq \log 2 + \log(x - 1)$.

Floor, ceiling, fractional

1. (11NE5) Solve for $x : 2 \lfloor x \rfloor = x + 2\{x\}$.
2. (14ND5) Solve for $x : 2x(x - \lfloor x \rfloor) = \lfloor x \rfloor^2$.
3. (13AI17) The number x is chosen randomly from the interval $(0, 1]$. Define $y = \lceil \log_4 x \rceil$. Find the sum of the lengths of all subintervals of $(0, 1]$ for which y is odd.

Value-finding

1. (11QI6) Given $f(1) = 5, f(x + 1) = 2f(x) + 1$, find $f(7) - f(0)$.
2. (13QII8) Suppose $f : \mathbb{R}^* \to \mathbb{R}^*$ and $f(a) + \dfrac{1}{f(b)} = f\left(\dfrac{1}{a}\right) + f(b)$. Find all possible values of $f(1) - f(-1)$.
3. (16NE2) A function $f(x)$ satisfies $(2 - x)f(x) - 2f(3 - x) = -x^3 + 5x - 18$ for all real x. Find $f(0)$.

Cauchy functional equation

1. (14QII5) Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x + y) = f(x)f(y), f(xy) = f(x) + f(y)$, for all $x, y \in \mathbb{R}$. Find $f(\pi^{2013})$.
2. (9QIII3) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $f(a + b) = f(a) + f(b)$. Given $f(2008) = 3012$, find $f(2009)$.
3. (13NE6) Given $f : \mathbb{R} \to \mathbb{R}$ and $f(a + b) = f(a)f(b)$. If $f(4) = 625$, what is $3f(-2)$?

Other functional equations

1. (10NE5) Given $f : \mathbb{R} \to \mathbb{R}^*$, and for all $x, y \in \mathbb{R}, f(x - y) = 2009f(x)f(y)$, find $f(\sqrt{2009})$.
2. (14QIII2) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 1$ and $f(2xy - 1) = f(x)f(y) - f(x) - 2y - 1$.
3. (16QIII2) If $f : \mathbb{R} \to \mathbb{R}, f(5) = 3$ and $f(4xy) = 2y[f(x + y) + f(x - y)]$, find $f(2015)$.
4. (10N3) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $x + f(x) + 2f\left(\dfrac{x + 2009}{x - 1}\right) = 2010$, for all $x \in \mathbb{R}$.
5. (11N4) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) + xf(x) = 1$, for all $x \in \mathbb{R}$.

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Session 2: Trigonometry
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September 23, 2016

Circular functions

1. (11QI8) Find the sum $\cos 1^\circ + \cos 3^\circ + \cos 5^\circ + \cdots + \cos 177^\circ + \cos 179^\circ$.
2. (13QI15) Find the value of $\sin\theta$ if the terminal side of θ lies on $5y - 3x = 0$ and θ is in the first quadrant.
3. (11AI14) The line from the origin to the point $(1, \tan 75^\circ)$ intersects the unit circle at P. Find the slope of the tangent line to the circle at P.
4. (11AI11) Find the sum of the coefficients of the polynomial $\cos(2\cos^{-1}(1-x^2))$.

Identities

1. (11QII5) Find the value of $\cos 15^\circ$.
2. (14QII6) Evaluate $\log_2 \sin(\pi/8) + \log_2 \cos(15\pi/8)$.
3. (16NE9) If $\tan x + \tan y = 5$ and $\tan(x+y) = 10$, find $\cot^2 x + \cot^2 y$.
4. (15AI4) Find the numerical value of $(1 - \cot 37^\circ)(1 - \cot 8^\circ)$.
5. (16NA1) Find the value of $\cot(\cot^{-1} 2 + \cot^{-1} 3 + \cot^{-1} 4 + \cot^{-1} 5)$.
6. (16AI6) Evaluate $\displaystyle\prod_{\theta=1}^{89}(\tan\theta^\circ \cos 1^\circ + \sin 1^\circ)$.
7. (13AI14) Given that $\tan\alpha + \cot\alpha = 4$, find $\sqrt{\sec^2\alpha + \csc^2\alpha - \frac{1}{2}\sec\alpha\csc\alpha}$.

Equations

1. (13QI11) If $2\sin(3x) = a\cos(3x + c)$, find all values of ac.
2. (13QI10) How many solutions has $\sin 2\theta - \cos 2\theta = \sqrt{6}/2$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$?
3. (10NA9) If $0 < \theta < \pi/2$ and $1 + \sin\theta = 2\cos\theta$, determine the numerical value of $\sin\theta$.
4. (13NE13) Find the solution set of the equation $\dfrac{\sec^2 x - 6\tan x + 7}{\sec^2 x - 5} = 2$.
5. (10ND4) Find the only value of x in $(-\pi/2, 0)$ that satisfies $\dfrac{\sqrt{3}}{\sin x} + \dfrac{1}{\cos x} = 4$.
6. (16AI13) Find all real numbers a and b so that for all real numbers x,
$$2\cos^2\left(x + \frac{b}{2}\right) - 2\sin\left(ax - \frac{\pi}{2}\right)\cos\left(ax - \frac{\pi}{2}\right) = 1.$$
7. (14AI12) Suppose $\alpha, \beta \in (0, \pi/2)$. If $\tan\beta = \dfrac{\cot\alpha - 1}{\cot\alpha + 1}$, find $\alpha + \beta$.
8. (14ND3) Find all $0 \le \theta \le 2\pi$ satisfying $\sqrt{\dfrac{1}{2} + \dfrac{1}{2}\sqrt{\dfrac{1}{2} + \dfrac{1}{2}\sqrt{\dfrac{1}{2} + \dfrac{1}{2}\cos 8\theta}}} = \cos\theta$.

Triangle laws

1. (16NE5) In right triangle $ABC, \angle ACB = 90^\circ$ and $AC = BC = 1$. Point D is on AB such that $\angle DCB = 30^\circ$. Find the area of $\triangle ADC$.
2. (13NE11) In $\triangle ABC, \angle A = 60^\circ, \angle B = 45^\circ$, and $AC = \sqrt{2}$. Find the area of the triangle.
3. (10QIII5) Let M be the midpoint of side BC of triangle ABC. Suppose that $AB = 4, AM = 1$. Determine the smallest possible measure of $\angle BAC$.
4. (13AI9) Consider an acute triangle with angles α, β, γ opposite the sides a, b, c respectively. If $\sin\alpha = \dfrac{3}{5}$ and $\cos\beta = \dfrac{5}{13}$, evaluate $\dfrac{a^2 + b^2 - c^2}{ab}$.
5. (15AII3) Points A, M, N and B are collinear, in that order, and $AM = 4, MN = 2, NB = 3$. If point C is not collinear with these four points, and $AC = 6$, prove that CN bisects $\angle BCM$.
6. (11AII2) Denote by a, b, c the sides of a triangle opposite angles α, β, γ, respectively. If $\alpha = 60^\circ$, prove that $a^2 = \dfrac{a^3 + b^3 + c^3}{a + b + c}$.

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Session 3: Number theory

compiled by Carl Joshua Quines

September 28, 2016

Ad hoc

1. (14QII8) How many trailing zeroes does 126! have when written in decimal notation?
2. (14AI7) What is the largest positive integer k such that 27! is divisible by 2^k?
3. (14NE2) What is the smallest number of integers that need to be selected from $\{1, 2, \ldots, 50\}$ to guarantee that two of the selected numbers are relatively prime?
4. (16QII4) How many positive integers n make the expression $7^n + 7^3 + 2 \cdot 7^2$ a perfect square?
5. (11ND2) Find all positive integers n make the expression $2^8 + 2^{11} + 2^n$ a perfect square.
6. (14NA7) What is the largest positive integer $abcdef$ that can be formed from the digits $1, 2, 3, 4, 5, 6$, each used exactly once, if $abcdef$ is divisible by 6, $abcde$ is divisible by 5, $abcd$ is divisible by 4, abc is divisible by 3, and ab is divisible by 2?
7. (14NA8) Let N be the smallest integer such that the quotient $\dfrac{10!}{N}$ is odd. If x and y are nonnegative numbers such that $2x + y = N$, what is the maximum value of x^2y^2?
8. (16AI14) Let P be the product of all prime numbers less than 90. Find the largest integer N so that for each $n \in \{2, 3, 4, \ldots, N\}$, the number $P + n$ has a prime factor less than 90.

Factors

1. (16QI7) What is the fifth largest divisor of the number $2,015,000,000$?
2. (11NA9) What is the sum of all the even positive divisors of 1152?
3. (15AI1) What is the fourth smallest positive integer having exactly 4 positive integer divisors, including 1 and itself?
4. (13NE4) If $25 \cdot 9^{2x} = 15^y$ is an equality of integers, what is the value of x?
5. (14QI14) How many factors of 7^{9999} are greater than $1,000,000$?
6. (13QI9) Determine the number of factors of $5^x + 2 \cdot 5^{x+1}$.
7. (14AI10) Let p and q be positive integers such that $pq = 2^3 \cdot 5^5 \cdot 7^2 \cdot 11$ and $\dfrac{p}{q} = 2 \cdot 5 \cdot 7^2 \cdot 11$. Find the number of positive integer divisors of p.
8. (11QIII3) Let $n = 2^{31}3^{19}$. How many positive divisors of n^2 are less than n but do not divide n?
9. (16AI7) Find the sum of all the prime factors of $27,000,001$.
10. (9AI16) Give the prime factorization of $3^{20} + 3^{19} - 12$.
11. (16NA10) Let m be the product of all positive integral divisors of $360,000$. Suppose the prime factors of m are $p_1, p_2, \ldots, p_k$ for some positive integer k, and $m = p_1^{e_1} p_2^{e_2} \cdot \cdots \cdot p_k^{e_k}$, for some positive integers $e_1, e_2, \ldots, e_k$. Find $e_1 + e_2 + \cdots + e_k$.
12. (12N2) Let f be a polynomial function with integer coefficients and p be a prime number. Suppose that there are at least four distinct integers satisfying $f(x) = p$. Show that f does not have integer zeros.

Divisibility

1. (9QII7) How many values of n for which n and $\dfrac{n+3}{n-1}$ are both integers?
2. (15AI11) Find all integer values of n that will make $\dfrac{6n^3-n^2+2n+32}{3n+1}$ an integer.
3. (15AII2) What is the greatest common factor of all integers of the form p^4-1, where p is a prime number greater than 5?
4. (14AII2) Let a, b, and c be positive integers such that $\dfrac{a\sqrt{2013}+b}{b\sqrt{2013}+c}$ is a rational number. Show that $\dfrac{a^2+b^2+c^2}{a+b+c}$ and $\dfrac{a^3-2b^3+c^3}{a+b+c}$ are both integers.
5. (11ND3) How many positive integer pairs (x, y) are solutions to the equation $\dfrac{xy}{x+y}=1000$?
6. (16N2) Prove that the arithmetic sequence $5, 11, 17, 23, 29, \ldots$ contains infinitely many primes.

Diophantine equations

1. (9QI5) How many ordered pairs (x, y) of positive integers satisfy $2x+5y=100$?
2. (16NE12) Find all values of integers x and y satisfying $2^{3x}+5^{3y}=189$.
3. (13NE3) Find the values of a such that the system $x+2y=a+6, 2x-y=25-2a$ has a positive integer pair solution (x, y).
4. (14QII3) If $2xy+y=43+2x$ for positive integers x, y, find the largest value of $x+y$.
5. (14NE4) Find positive integers a, b, c such that $a+b+ab=15, b+c+bc=99$ and $c+a+ca=399$.
6. (11NA5) Find all nonnegative integer solutions of the system $5x+7y+5z=37, 6x-y-10z=3$.
7. (13NE8) If $7x+4y=5$ and x and y are integers, find the value of $\lfloor y/x \rfloor$.
8. (14NE9) How many triples (x, y, z) of positive integers satisfy the equation $x^{y^z}y^{z^x}z^{x^y}=3xyz$?
9. (13N1) Determine, with proof, the least positive integer n for which there exists n distinct positive integers $x_1, x_2, x_3, \ldots, x_n$ such that

$$\left(1-\frac{1}{x_1}\right)\left(1-\frac{1}{x_2}\right)\left(1-\frac{1}{x_3}\right)\cdots\left(1-\frac{1}{x_n}\right)=\frac{15}{2013}.$$

Modulo

1. (11AI4) Find the last 2 nonzero digits of 16!.
2. (16QI12) When $n+5$ is divided by 4, the remainder is 3. When $n+4$ is divided by 5, the remainder is 5. What is the remainder when $n+6$ is divided by 20?
3. (16NE7) Let n be a positive integer greater than 1. If $2n$ is divided by 3, the remainder 2. If $3n$ is divided by 4, the remainder is 3. If $4n$ is divided by 5, the remainder is 4. If $5n$ is divided by 6, the remainder is 5. What is the least possible value of n?
4. (14NA2) Find the remainder when $3!^{5!^{7!^{\cdots^{2013!}}}}$ is divided by 11.
5. (15AI17) What is the remainder when $16^{15}-8^{15}-4^{15}-2^{15}-1^{15}$ is divided by 96?
6. (16AII1) The 6-digit number $739ABC$ is divisible by $7, 8$, and 9. What values can A, B, and C take?
7. (13N4) Let a, p, and q be positive integers with $p \leq q$. Prove that if one of the numbers a^p and a^q is divisible by p, then the other number must also be divisible by p.

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Session 4: Combinatorics 1

compiled by Carl Joshua Quines

September 30, 2016

Ad hoc

1. (16QII9) Find the 2015th digit in $122333444455555\ldots$
2. (15AI14) In how many ways can Alex, Billy and Charles split 7 identical marbles among themselves so that no two have the same number of marbles? It is possible for someone not to get any marbles.
3. (15AI15) In a word finding game, a player tries to find a word in a 12×12 array of letters by looking at blocks of adjacent letters are arranged horizontally, arranged vertically or arranged diagonally. How many such 3-letter blocks are there in a given 12×12 array of letters?
4. (10NA2) The positive integers are grouped as follows: $A_1 = \{1\}, A_2 = \{2, 3, 4\}, A_3 = \{5, 6, 7, 8, 9\}$ and so on. In which group does 2009 belong to?
5. (13NA1) How many pairs of diagonals on the surface of a rectangular prism are skew?
6. (13AII3) Let $v(X)$ be the sum of the elements of a set X. Calculate the sum of all numbers $v(X)$ where X ranges over all non-empty subsets of the set $\{1, 2, 3, \ldots, 16\}$.
7. (16ND1) The irrational number $0.123456789101112\ldots$ is formed by concatenating, in increasing order, all the positive integers. Find the sum of the first 2016 digits of this number after the decimal point.
8. (16ND3) In an $n \times n$ checkerboard, the rows are numbered 1 to n from top to bottom, and the columns are numbered 1 to n from left to right. Chips are to be placed on this board so that each square has a number of chips equal to the absolute value of the difference of the row and column numbers. If the total number of chips placed on the board is 2660, find n.
9. (16AI20) Let s_n be the sum of the digits of a natural number n. Find the smallest value of $\dfrac{n}{s_n}$ if n is a four-digit number.
10. (9ND5) The vertices of a cube are each colored by either black or white. Two colorings of the cube are said to be *geometrically the same* if one can be obtained from the other by rotating the cube. In how many geometrically different ways can the coloring be done?

Inclusion-Exclusion

1. (13AI15) There are 100 people in a room. 60 of them claim to be good at math, but only 50 are actually good at math. If 30 of them correctly deny that they are good at math, how many people are good at math but refuse to admit it?
2. (16QI6) How many positive integers less than or equal to 2015 are divisible by 3, but are neither divisible by 5 nor 7?
3. (9AI17) How many integers between 2 and 10000 do not share a prime factor with 10000?
4. (16QII8) How many integers x are there, where $100 \leq x \leq 2015$, and x is divisible by 3 or 8, but not by 6?
5. (10QII3) How many distinct natural numbers less than 1000 are multiples of $10, 15, 35$ or 55?

Permutations

1. (14NE13) How many three digit positive integers are there, the sum of whose digits is a perfect cube?
2. (16QI11) In how many ways can the letters of the word QUALIFYING be arranged such that the vowels are all in alphabetical order?
3. (15AI8) How many ways can 6 boys and 6 girls be seated in a circle so that no two boys sit next to each other?
4. (16AI15) In how many ways can the letters of the word ALGEBRA be arranged if the order of the vowels must remain unchanged?
5. (14AI13) How many positive integers, not having the digit 1, can be formed if the product of all its digits is to be 33750?
6. (14QI11) If all the words obtained from permuting the letters of the word $SMART$ are arranged alphabetically, what is the rank of the word SMART?
7. (14ND4) In how many ways can the letters in the word PHILLIP be arranged so that both of the strings PHI or ILL do not appear?
8. (11QII10) In how many ways can the letters of the word MURMUR be arranged without letting two letters which are the same be adjacent?

Combinations

1. (14NE5) A set S has n elements. There are exactly 57 subsets of S with two or more elements. How many elements does S have?
2. (10QI6) How many ways can three distinct numbers be selected from the set $\{1, 2, 3, \ldots, 9\}$ if the product of these numbers is divisible by 21?
3. (13AI12) Six boy-girl pairs are to be formed from a group of six boys and six girls. In how mnay ways can this be done?

Balls and urns

1. (16QI14) Mary wants to give 15 cookies to Amy, Bob and Charlie. How many ways can she distribute the cookies to them such that Amy must receive at least 5 cookies while Bob and Charlie must each receive at least 1?
2. (16AI9) How many ways can you place 10 identical balls in 3 baskets of different colors if it is possible for a basket to be empty?
3. (16NE11) How many solutions does the equation $x + y + z = 2016$ have, where x, y and z are integers with $x > 1000, y > 600$ and $z > 400$?
4. (10NE7) How many ways can you choose four integers from the set $\{1, 2, 3, \ldots, 10\}$ so that no two of them are consecutive?
5. (13QIII4) In how many ways can one select five books from a row of twelve books so that no two adjacent books are chosen?
6. (16QII5) The numbers $1, 2, \ldots, 12$ have been arranged in a circle. In how many ways can five numbers be chosen from this arrangement so that no two adjacent numbers are selected?
7. (16QIII4) Let $N = \{0, 1, 2, 3, \ldots\}$. Find the cardinality of the set
$$\{(a, b, c, d, e) \in N^5 : 0 \leq a + b \leq 2, 0 \leq a + b + c + d + e \leq 4\}.$$

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Session 5: Algebra 2

compiled by Carl Joshua Quines

October 5, 2016

Equations

1. (13QI7) Sixty men working on a job have done 1/3 of the work in 18 days. The project is behind schedule and must be accomplished in the next twelve days. How many more workers need to be hired?

2. (14AI14) Solve the equation $(2-x^2)^{x^2-3\sqrt{2}x+4}=1$.

3. (11NE4) If $\dfrac{x-a-b}{c}+\dfrac{x-b-c}{a}+\dfrac{x-c-a}{b}=3$, where $a,b,c\in\mathbb{R}^+$, find x in terms of a,b and c.

4. (14AI19) Find the values of x in $(0,\pi)$ that satisfy the equation

$$(\sqrt{2014}-\sqrt{2013})^{\tan^2 x}+(\sqrt{2014}+\sqrt{2013})^{-\tan^2 x}=2(\sqrt{2014}-\sqrt{2013})^3$$

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5. (14QI7) For which m does the equation $\dfrac{x-1}{x-2}=\dfrac{x-m}{x-6}$ have no solution in x?

6. (16AI3) Determine all values of $k\in\mathbb{R}$ for which $\dfrac{4(2015^x)-2015^{-x}}{2015^x-3(2015^{-x})}=k$ admits a real solution.

7. (11ND4) Give three real roots of $\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$.

8. (13NE5) Find the solution set of the equation $(\sqrt{2}-1)^x+8(\sqrt{2}+1)^x=9$.

Systems of equations

1. (8NE1) If $wxy=10, wyz=5, wxz=45, xyz=12$, what is $w+y$?

2. (16AI12) Find all real solutions to the system of equations $x(y-1)+y(x+1)=6, (x-1)(y+1)=1$.

3. (13QI14) If (a,b) is the solution of $\sqrt{x+y}+\sqrt{x-y}=4, x^2-y^2=9$, then find the value of $\dfrac{ab}{a+b}$.

4. (14NA5) Suppose that $w+4x+9y+16z=6, 4w+9x+16y+25z=7, 9w+16x+25y+36z=12$. Find $w+x+y+z$.

5. (10NA1) The nonzero numbers x,y,z satisfy $xy=2(x+y), yz=4(y+z), xz=8(x+z)$. Solve for x.

Complex numbers

1. (11AI8) Find all complex numbers x satisfying $x^3+x^2+x+1=0$.

2. (15AI3) Simplify the expression $\left(1+\dfrac{1}{i}+\dfrac{1}{i^2}+\cdots+\dfrac{1}{i^{2014}}\right)^2$.

3. (13AI1) Find all complex numbers z such that $\dfrac{z^4+1}{z^4-1}=\dfrac{i}{\sqrt{3}}$.

4. (13AI3) If $z^3-1=0$ and $z\neq 1$, find the value of $z+\dfrac{1}{z}+4$.

Polynomials

1. (13NE7) Let $P(x) = ax^7 + bx^3 + cx - 5$, where a, b, and c are constants. If $P(-7) = 7$, what is $P(7)$?
2. (13NE1) In solving a problem that leads to a quadratic equation, one student made a mistake in the constant term only, obtaining the roots 8 and 2, while another student made a mistake in the coefficient of the first degree term, obtaining the roots -9 and -1. What was the original equation?
3. (13AI5) Consider a function $f(x) = ax^2 + bx + c, a > 0$, with two distinct roots a distance p apart. By how much, in terms of a, b, c, should the function be translated downwards so that the distance between the roots becomes $2p$?
4. (15AI9) Two numbers p and q are both chosen randomly and independently of each other from the interval $[-2, 2]$. Find the probability that $4x^2 + 4px + 1 - q^2 = 0$ has imaginary roots.
5. (9NA4) What is the coefficient of x^5 in the expansion of $(2 - x + x^2)^4$?
6. (13AII2) The quartic polynomial $P(x)$ satisfies $P(1) = 0$ and attains its maximum value of 3 at both $x = 2$ and $x = 3$. Find $P(5)$.
7. (10NA8) When $(x^2 + 2x + 2)^{2009} + (x^2 - 3x - 3)^{2009}$ is expanded, what is the sum of the coefficients of the terms with odd exponents of x?
8. (16QIII5) How many terms are there when the expression of $(x+y+z)^{2015} + (x-y-z)^{2015}$ is expanded and simplified?

Polynomial factors

1. (10NE11) Find the values of a and b such that $ax^4 + bx^2 + 1$ is divisible by $x^2 - x - 2$.
2. (13NE9) If $x^2 + 2x + 5$ is a factor of $x^4 + ax^2 + b$, find the sum $a + b$.
3. (13QII9) Factorize $(r - s)^3 + (s - t)^3 + (t - r)^3$.
4. (16QII6) How many (nonconstant) polynomial factors with leading coefficient 1, with the other coefficients possibly complex, does $x^{2015} + 18$?
5. (11QIII2) Find all polynomials $p(x)$ where $xp(x - 1) = (x - 5)p(x)$ and $p(6) = 5!$.

Remainder theorem

1. (14AI11) Let r be some real constant, and $P(x)$ a polynomial which has remainder 2 when divided by $x - r$, and remainder $-2x^2 - 3x + 4$ when divided by $(2x^2 + 7x - 4)(x - r)$. Find all values of r.
2. (11AI5) Let $f(x)$ be a cubic polynomial. If $f(x)$ is divided by $2x + 3$, the remainder is 4, while if it is divided by $3x + 4$, the remainder is 5. What will be the remainder when $f(x)$ is divided by $6x^2 + 17x + 22$?
3. (9NA7) Let $P(x)$ be a polynomial, that, when divided by $x-19$, has the remainder 99, and when divided by $x - 99$, has the remainder 19. What is the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$?

Root-finding

1. (16QII2) What is the difference between the largest and smallest real zeros of the function $f(x) = 2x^4 - 7x^3 + 2x^2 + 7x + 2$?
2. (13NA8) There are values of m for which $x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$ has equal roots. What are these equal roots?
3. (13AI11) Let f be a polynomial function that satisfies $f(x - 5) = -3x^2 + 45x - 108$. Find the roots of $f(x)$.

4. (14AII3) If p is a real constant such that the roots of the equation $x^3 - 6px^2 + 5px + 88 = 0$ form an arithmetic sequence, find p.

5. (11NE11) $x^3 + kx - 128 = 0$ has a root of multiplicity 2. Find k.

6. (11AI17) Find all real numbers a such that $x^3 + ax^2 - 3x - 2$ has exactly two distinct real zeros.

Vieta's

1. (16NE4) There are two distinct real numbers which are larger than their reciprocals by 2. Find the product of these numbers.

2. (16NA4) Let $f(x)$ be a polynomial function of degree 2016 whose 2016 zeros have a sum of S. Find the sum of the 2016 zeros of $f(2x-3)$ in terms of S.

3. (9QII1) The roots of the quadratic equation $x^2 - 51x + k = 0$ differ by 75, where k is a real number. Determine the sums of the squares of the roots.

4. (11NE10) Find a quadratic polynomial with integer coefficients whose roots are the reciprocals of $x^2 + 4x + 8 = 0$.

5. (9NE13) Find the sum of the reciprocals of the roots of $4x^4 - 3x^3 - x^2 + 2x - 6 = 0$.

6. (13QII5) The equation $x^2 - bx + c$ has two roots p and q. If the product pq is to be maximum, what value of b will make $b + c$ minimum?

7. (14NE11) Suppose a, b, c are the roots of $x^3 - 4x + 1 = 0$. Find the value of $\dfrac{a^2bc}{a^3+1} + \dfrac{ab^2c}{b^3+1} + \dfrac{abc^2}{c^3+1}$.

Coordinate plane

1. (11QI4) For what values of a does the system $x^2 - y^2 = 0, (x-a)^2 + y^2 = 0$ have a unique solution?

2. (10NE6) If the parabola $y + 1 = x^2$ is rotated clockwise by 90° about its focus, what will be the new coordinates of its vertex?

3. (14AI8) For what real values of p will the graph of the parabola $y = x^2 - 2px + p + 1$ be on or above that of the line $y = -12x + 5$?

4. (14QII4) Let (a, b) and (c, d) be the two distinct points of intersection of circles C_1 and C_2. The circle C_1 is centered at the origin and passes through $P(16, 16)$, while the circle C_2 is centered at P and passes through the origin. Find $a + b + c + d$.

5. (13NA9) Let $A(-3, 0), B(3, 0), C(0, 5)$ and $D(0, -5)$. How many points $P(x, y)$ on the plane satisfy both $PA + PB = 10$ and $|PC - PD| = 6$?

6. (11QII7) A line with y-intercept 5 and positive slope is drawn such that the line intersects $x^2 + y^2 = 9$. What is the least slope of such a line?

7. (16AI17) Find the area of the region bounded by the graph of $|x| + |y| = \dfrac{1}{4}|x + 15|$.

8. (13AI4) Find the equation of the line that contains the point $(1, 0)$, that is of least positive slope, and that does not intersect the curve $4x^2 - y^2 - 8x = 12$.

9. (13AI6) Find the equation of the circle, in the form $(x-h)^2 + (y-k)^2 = r^2$, inscribed in a triangle whose vertices are located at the points $(-2, 1), (2, 5), (5, 2)$.

10. (13NE12) What is the length of the shortest path that begins at the point $(-3, 7)$, touches the x-axis, and then ends at a point on the circle $(x-5)^2 + (y-8)^2 = 25$?

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Session 6: Combinatorics 2
compiled by Carl Joshua Quines
October 7, 2016

Random variable

1. (14QI3) What is the probability of getting a sum of 10 when rolling three fair six-sided dice?
2. (13QII7) A fair die is thrown three times. What is the probability that the largest outcome of the three throws is a 3?
3. (11AII1) Sherlock and Mycroft play a game which involves flipping a single fair coin. The coin is flipped repeatedly until one person wins. Sherlock wins if the sequence TTT shows up first while Mycroft wins if the sequence HTT shows up first. Who among the two has a higher probability of winning?
4. (16ND5) The faces of a 12-sided die are numbered $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$, and 12 such that the sum of the numbers on opposite faces is 13. The die is meticulously carved so that it is biased: the probability of obtaining a particular face F is greater than $1/12$, the probability of obtaining the face opposite F is less than $1/12$, while the probability of obtaining any one of the other ten faces, is $1/12$. When two such dice are rolled, the probability of obtaining a sum of 13 is $29/384$. What is the probability of obtaining face F?

Random selection

1. (11QII2) Mica has six different colored crayons. She can use one ore more colors in her painting. What is the likelihood that she will use only her favorite color?
2. (16QI15) The exterior of a cube of side length 3 units is painted red, and is then divided into 27 unit cubes. If one of these cubes is randomly selected and then rolled, what is the probability that a red face comes up?
3. (11NE7) Let a, b, c be three, not necessarily distinct, numbers chosen randomly from the set $\{3, 4, 5, 6, 7, 8\}$. Find the probability that $ab + c$ is even.
4. (16NE6) An urn contains five red chips numbered 1 to 5, five blue chips numbered 1 to 5, and five white chips numbered 1 to 5. Two chips are drawn from this urn without replacement. What is the probability that they have either the same color or the same number?
5. (16NA8) In a certain school, there are 5000 students. Each student is assigned an ID number from 0001 to 5000. No two students can have the same ID number. If a student is selected uniformly at random, what is the probability that the ID number of the student does not contain any 2s among its digits?
6. (16NA6) Suppose that Ethan has four red chips and two white chips. He selects three chips at random and places them in urn 1, while the remaining chips are placed in urn 2. He lets his brother Josh draw one chip from each urn at random. What is the probability that the chips drawn by Josh are both red?
7. (14NA6) A bag contains 3 red and 3 green balls. Three balls are drawn at random from the bag and replaced with 3 white balls. Afterwards, another 3 balls are drawn at random from the bag. Find the probability that the color of the 3 balls in the second draw are all different.
8. (13NA7) Twenty-five people sit around a circular table. Three of them are chosen randomly. What is the probability that two of the three are sitting next to each other?
9. (12N1) A computer generates even integers half of the time and another computer generates even integers a third of the time. If a_i and b_i are the integers generated by the computers, respectively, at time i, show that the probability $a_1b_1 + \cdots + a_kb_k$ is even is $\dfrac{1}{2} + \dfrac{1}{2 \cdot 3^k}$.

Geometric probability

1. (14NE8) Given two distinct points A and B, a point P is randomly and uniformly chosen in the interior of segment AB. Let $r > 0$. Find the probability, in terms of r, that $\dfrac{AP}{BP} < r$.

2. (16QII1) A married couple has PhP $50,000$ in their joint account. In anticipation of their upcoming anniversary, they agreed to split-up evenly and buy each one a gift. However, they agreed that they must have enough money left such that when combined, a minimum bank balance of PhP $5,000$ is maintained. If the couple individually bought gifts without regard to how much the other's gift will cost, and each gift is randomly price between PhP 0 to PhP $25,000$, what is the probability that they will be able to maintain the minimum required balance after buying the gifts?

3. (11AI7) Find the probability of obtaining two numbers x and y in the interval $[0,1]$ such that $x^2 - 3xy + 2y^2 > 0$.

4. (16AI19) The amount 4.5 is split into two nonnegative real numbers uniformly at random. Then each number is rounded to its nearest integer. For instance, if 4.5 is split into $\sqrt{2}$ and $4.5 - \sqrt{2}$, then the resulting integers are 1 and 3, respectively. What is the probability that the two integers sum to 5?

Existence combinatorics

1. (10QII9) Seven distinct integers are chosen randomly from the set $\{1, 2, \ldots, 2009\}$. What is the probability that two of these integers have a difference that is a multiple of 6?

2. (13AI13) From the xy-plane, select five distinct points that have integer coordinates. Find the probability that there is a pair of points among the five whose midpoint has integer coordinates.

3. (9N3) Each point of a circle is colored either red or blue. Prove that there always exists an isosceles triangle inscribed in this circle such that all its vertices are colored the same. Does there always exist an equilateral triangle inscribed in this circle such that all its vertices are colored the same?

4. (10N4) There are 2008 blue, 2009 red, and 2010 yellow chips on a table. At each step, one chooses two chips of different colors, and recolor both of them using the third color. Can all chips be of the same color after some steps?

5. (10N5) Determine the smallest positive integer n such that for every choice of n integers, there exists at least two whose sum or difference is divisible by 2009.

6. (11N3) The 2011th prime number is 17483, and the next prime is 17489. Does there exist a sequence of 2011^{2011} consecutive positive integers that contains exactly 2011 prime numbers? Prove your answer.

7. (13N3) Let n be a positive integer. The numbers $1, 2, 3, \ldots, 2n$ are randomly assigned to $2n$ distinct points on a circle. To each chord joining two of these points, a value is assigned equal to the absolute value of the difference between the assigned numbers at its endpoints. Show that one can choose n pairwise non-intersecting chords such that the sum of the values assigned to them is n^2.

8. (12N5) There are exactly 120 Twitter subscribers from National Science High School. Statistics shows that each of 10 given celebrities have at least 85 followers from National Science High School. Prove that there must be two students such that each of the 10 celebrities is being followed in Twitter by at least one of these students.

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Session 7: Geometry 1

compiled by Carl Joshua Quines

October 12, 2016

Circles

1. (16QI5) The two internal tangents of two non-overlapping circles of radii 2 and 4 units intersect at right angles. What is the distance between the centers of the circles?

2. (16QI9) Triangles PQR and QRS, where $P \neq S$, are two right triangles sharing the hypotenuse QR. If C_1 is the circle passing through P, Q, and R, and C_2 is the circle passing through Q, R, and S, what can be said about $C_1 \cap C_2$?
   (a) $C_1 \cap C_2$ consists of exactly one point.
   (b) $C_1 \cap C_2$ consists of exactly two distinct points.
   (c) $C_1 \cap C_2$ is empty.
   (d) $C_1 \cap C_2$ consists of infinitely many points.

3. (14QI15) Let AB be a chord of circle C with radius 13. If the shortest distance of AB to point C is 5, what is the perimeter of ABC?

4. (15AI18) Segment CD is tangent to the circle with center O, at D. Point A is in the interior of the circle, and segment AC intersects the circle at B. If $OA = 2, AB = 4, BC = 3$ and $CD = 6$, find the length of segment OC.

5. (14AII1) Two circles C_1 and C_2 of radius 12 have their centers on each other. A is the center of C_1 and AB is a diameter of C_2. A smaller circle is constructed tangent to AB and the two given circles, externally to C_1 and internally to C_2. Find the radius of this smaller circle.

6. (16ND4) Cyclic quadrilateral $ABCD$ has $DA = BC = 2$ and $AB = 4$. If $CD > AB$ and the lines DA and BC intersect at an angle of $60°$, find the radius of the circumscribing circle.

Angles

1. (16QI10) In triangle ABC, BD is the angle bisector of $\angle ABC$ such that $AB = BD$. Point E is on AB such that $AE = AD$. If $\angle ACB = 36°$, find $\angle BDE$.

2. (16NE13) In parallelogram $ABCD, \angle BAD = 76°$. Side AD has midpoint P, and $\angle PBA = 52°$. Find $\angle PCD$.

3. (14QII10) In ABC, point D is on AC such that $AB = AD$ such that $\angle ABC - \angle ACB = 45°$. Find $\angle CBD$.

4. (15AI20) Trapezoid $ABCD$ has right angles at C and D, and $AD > BC$. Let E and F be points on AD and AB, respectively, such that $\angle BED$ and $\angle DFA$ are right angles. Let G be the point of intersection of the segments BE and DF. If $\angle CED = 58°$ and $\angle FDE = 41°$, what is $\angle GAB$?

5. (13ND2) Circles A and B are tangent to each other externally at E. The segment CG is tangent externally to both circles, with C on circle A and G on circle B. A point D is selected on circle A and F on circle B such that D, E, and F are collinear. The measure of minor arc CE is $102°$. Find the measure of $\angle DFG$.

Three-dimensional

1. (13QI4) A convex polyhedron has 30 faces and 62 edges. How many vertices does this polyhedron have?
2. (14NE15) Six matchsticks, each 1 unit long, are used to form a pyramid having equilateral triangles for its 4 faces. What is the volume of this pyramid?
3. (14AI2) What is the largest number of $7 \times 9 \times 11$ boxes that can fit inside a box of size $17 \times 37 \times 27$?
4. (16NA9) 120 unit cubes are put together to form a rectangular prism whose six faces are then painted. This leaves 24 unit cubes without any paint. What is the surface area of the prism?
5. (11QIII4) Four spheres, each of radius 1.5, are placed in a pile with three at the base and the other at the top. If each sphere touches the other three spheres, give the height of the pile.

Areas

1. (14NE3) A horse is tied outside a fenced triangular garden at one of the vertices. The triangular fence is equilateral with side length equal to 8 units. If the rope with which the horse is tied is 10 units long, find the area over which the horse can graze outside the fence assuming that the rope and the fence are strong enough to hold the animal.
2. (16NE10) Let $ABCD$ be a trapezoid with parallel sides AB and CD of lengths 6 units and 8 units, respectively. Let E be the point of intersection of the extensions of the nonparallel sides of the trapezoid. If the area of BEA is 60 square units, what is the area of BAD?
3. (14NE6) In rectangle $ABCD$, point E is chosen in the interior of AD, and point F is chosen in the interior of BC. Let AF and BE meet at G, and CE and DF at H. The following areas are known: $[AGB] = 9, [BGF] = 16, [CHF] = 11, [CHD] = 15$. Find $[EGFH]$.
4. (13QIII1) $ABCD$ is a trapezoid with $AB||CD, AB = 6$ and $CD = 15$. If the area of AED is 30, find the area of AEB.
5. (16AI8) A side of an equilateral triangle is the diameter of a circle. If the radius of the circle is 1, find the area of the region inside the triangle but outside the circle.
6. (16NA5) Square $ABCD$ with side length 2 units has M and N as the midpoints of AD and BC. Point P is the intersection of segments AN and BM, and lines CP and DP meet side AB at points Q and R. Find the sum of the areas of triangles AMP, BNP, CPD and PQR.
7. (14AI15) Rectangle $BRIM$ has $BR = 16$ and $BM = 18$. The points A and H are located on IM and BM, respectively, so that $MA = 6$ and $MH = 8$. If T is the intersection of BA and IH, find the area of quadrilateral $MATH$.
8. (14AI17) Trapezoid $ABCD$ has parallel sides AB and CD, with BC perpendicular to them. Suppose $AB = 13, BC = 16$ and $DC = 11$. Let E be the midpoint of AD and F the point on BC so that EF is perpendicular to AD. Find the area of quadrilateral $AEFB$.
9. (14AI20) The base AB of a triangular piece of paper ABC is 16 units long. The paper is folded down over the base, with the crease DE parallel to the base of the paper. The area of the triangle that projects below the base is 16% the area of ABC. What is the length of DE?
10. (11AI18) A circle with center C and radius r intersects the square $EFGH$ at H and at M, the midpoint of EF. If C, E and F are collinear and E lies between C and F, what is the area of the region outside the circle but inside the square in terms of r?
11. (13N2) Let P be a point in the interior of ABC. Extend AP, BP and CP to meet BC, AC and AB at D, E and F, respectively. If APF, BPD and CPE, have equal areas, prove that P is the centroid of ABC.

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Session 8: Algebra 3

compiled by Carl Joshua Quines

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Manipulation

1. (13NE10) If $\dfrac{x}{y}+\dfrac{y}{x}=4$ and $xy=3$, find the value of $xy(x+y)^2-2x^2y^2$.
2. (14NE1) If $x=\sqrt{2013-yz}, y=\sqrt{2014-zx}$ and $z=\sqrt{2015-xy}$, find $(x+y)^2+(y+z)^2+(z+x)^2$.
3. (14QIII4) If $m^3-12mn^2=40$ and $4n^3-3m^2n=10$, find m^2+4n^2.
4. (13AII1) If $x+y+xy=1$, where x, y are nonzero real numbers, find the value of $xy+\dfrac{1}{xy}-\dfrac{y}{x}-\dfrac{x}{y}$.
5. (14QIII3) If $\dfrac{a}{a^2+1}=\dfrac{1}{3}$, determine $\dfrac{a^3}{a^6+a^5+a^4+a^3+a^2+a+1}$.
6. (14QII9) It is known that $1+\dfrac{1}{4}+\dfrac{1}{9}+\dfrac{1}{16}+\cdots=\dfrac{\pi^2}{6}$. Find the sum $1+\dfrac{1}{9}+\dfrac{1}{25}+\dfrac{1}{49}+\cdots$.
7. (14AI18) Let x be a real number so that $x+\dfrac{1}{x}=3$. Find the last two digits of $x^{2^{2013}}+\dfrac{1}{x^{2^{2013}}}$.
8. (13ND5) Let $x=cy+bz, y=az+cx, z=bx+ay$. Find $\dfrac{(x-y)(y-z)(z-x)}{xyz}$ in terms of a, b, and c.

Surds

1. (15AI6) Rationalize the denominator of $\dfrac{6}{\sqrt[3]{4}+\sqrt[3]{16}+\sqrt[3]{64}}$ and simplify.
2. (13QII10) If $36-4\sqrt{2}-6\sqrt{3}+12\sqrt{6}=(a\sqrt{2}+b\sqrt{3}+c)^2$, find the value of $a^2+b^2+c^2$.
3. (14ND2) Solve for x: $\sqrt{x+\sqrt{3x+6}}+\sqrt{x-\sqrt{3x+6}}=6$.
4. (13NA6) If $\sqrt[3]{a+\sqrt{b}}=12+\sqrt{5}$, find the value of $\sqrt[3]{a-\sqrt{b}}$.
5. (11NA8) Let $a=\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ and $b=\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$. Find the value of $a^4+b^4+(a+b)^4$.
6. (11AII3) Show that $\sqrt[n]{2}-1\le\sqrt{\dfrac{2}{n(n-1)}}$ for all positive integers $n\ge 2$.

Sequences

1. (14AI4) The sequence $2, 3, 5, 6, 7, 8, 10, 11, \ldots$ is an enumeration of the positive integers that are not perfect squares. What is the 150th term of this sequence?
2. (14QIII1) Let $\{a_n\}$ be a sequence such that the average of the first and second terms is 1, the average of the second and third terms is 2, the average of the third and fourth terms is 3, and so on. Find the average of the first and hundredth terms.
3. (11QII3) If $b_1=\dfrac{1}{3}$ and $b_{n+1}=\dfrac{1-b_n}{1+b_n}$ for $n\ge 2$, find $b_{2010}-b_{2009}$.

4. (13AI8) Let $3x, 4y, 5z$ form a geometric sequence while $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ form an arithmetic sequence. Find the value of $\frac{x}{z} + \frac{z}{x}$.

5. (16AII2) The numbers from 1 to 36 are written in a counterclockwise spiral as follows:

$$\begin{array}{ccccc} \boxed{13} & 12 & 11 & 10 & 25 \\ 14 & \boxed{3} & 2 & 9 & 24 \\ 15 & 4 & \boxed{1} & 8 & 23 \\ 16 & 5 & 6 & \boxed{7} & 22 \\ 17 & 18 & 19 & 20 & \boxed{21} \end{array}$$

In the figure above, all the terms on the diagonal beginning from the upper left corner have been enclosed in a box, and these entries sum up to 45.

Suppose this spiral is continued all the way until 2015, leaving an incomplete square. Find the sum of all the terms on the diagonal beginning from the upper left corner of the resulting (incomplete) square.

6. (14NE10) In the sequence $\{a_n\}, a_1 = 1, a_{n+1} = \dfrac{a_n}{1 + ca_n}$ for some constant c. If $a_{12} = \dfrac{1}{2014}$, find c.

7. (9N1) In the sequence $\{a_n\}, n(n+1)a_{n+1} + (n-2)a_{n-1} = n(n-1)a_n$ for every positive integer n, where $a_0 = a_1 = 1$. Calculate the sum $\dfrac{a_0}{a_1} + \dfrac{a_1}{a_2} + \cdots + \dfrac{a_{2008}}{a_{2009}}$.

Series

1. (10QIII4) A sequence of consecutive positive integers beginning with 1 is written on the blackboard. A student came along and erased one number. The average of the remaining numbers is $35\frac{7}{17}$. What number was erased?

2. (14NA3) Let $n \leq m$ be positive integers such that the first n numbers in $\{1, 2, 3, \ldots, m\}$ and the last $m - n$ numbers in the same sequence have the same sum 3570. Find m.

3. (14QII7) If the sum of the infinite geometric series $\dfrac{a}{b} + \dfrac{a}{b^2} + \dfrac{a}{b^3} + \cdots$ is 4, then what is the sum of $\dfrac{a}{a+b} + \dfrac{a}{(a+b)^2} + \dfrac{a}{(a+b)^3} + \cdots$?

4. (13NA4) Find the value of the infinite sum $1 + 1 + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^4 + \cdots$

5. (11NE8) Evaluate $\dfrac{1}{3} + \dfrac{1}{15} + \dfrac{1}{35} + \dfrac{1}{63} + \dfrac{1}{99} + \dfrac{1}{143} + \dfrac{1}{195}$.

6. (16AI10) Find the largest number N so that $\displaystyle\sum_{n=5}^{N} \frac{1}{n(n-2)} < \frac{1}{4}$.

7. (16QII10) Find the sum of $\displaystyle\sum_{i=1}^{2015} \left\lfloor \frac{\sqrt{i}}{10} \right\rfloor$.

8. (13AI7) Define $f(x) = \dfrac{a^x}{a^x + \sqrt{a}}$ for any $a > 0$. Evaluate $\displaystyle\sum_{i=1}^{2012} f\left(\frac{i}{2013}\right)$.

9. (11NA6) Find the sum $\displaystyle\sum_{k=1}^{19} k\binom{19}{k}$.

Inequalities

1. (13AI18) Find all k so that the inequality $k(x^2+6x-k)(x^2+x-12)>0$ has solution set $(-4,3)$.
2. (16NE14) Find the smallest k such that for all real x, y, z, $(x^2+y^2+z^2)^2 \le k(x^4+y^4+z^4)$.
3. (14NE14) Find the greatest k such that for any positive real a_1, a_2, a_3, a_4, a_5, with sum S, we have: $(S-a_1)(S-a_2)(S-a_3)(S-a_4)(S-a_5) \ge k(a_1a_2a_3a_4a_5)$.
4. (12N3) If $ab>0$ and $0<x<\dfrac{\pi}{2}$, prove that
$$\left(1+\frac{a^2}{\sin x}\right)+\left(1+\frac{b^2}{\cos x}\right) \ge \frac{(1+\sqrt{2}ab)^2 \sin 2x}{2}.$$
5. (9N4) Let k be a positive integer such that $\dfrac{1}{k+a}+\dfrac{1}{k+b}+\dfrac{1}{k+c} \le 1$ for any positive real numbers a, b, c with $abc=1$. Find the minimum value of k.

Single-variable extrema

1. (13QIII2) Find the maximum of $y=(7-x)^4(2+x)^5$ when x lies strictly between -2 and 7.
2. (14NE7) Find the maximum of $4x-x^4-1$.
3. (14NA10) Find the maximum of $\sqrt{(x-4)^2+(x^3-2)^2}-\sqrt{(x-2)^2+(x^3+4)^2}$.
4. (16NA2) Find the minimum of x^2+4y^2-2x, where x, y are reals that satisfy $2x+8y=3$.
5. (13ND4) Find the minimum of $a^6+a^4-a^3-a+1$.

Multi-variable extrema

1. (10QI7) If $|2x-3| \le 5$ and $|5-2y| \le 3$ find the minimum of $x-y$.
2. (15AI16) Find the maximum of $\displaystyle\sum_{i=1}^{2014}(\sin\theta_i)(\cos\theta_{i+1})$, where $\theta_1=\theta_{2015}$.
3. (8ND2) If a and b are positive real numbers, what is the minimum of $\sqrt{a+b}\left(\dfrac{1}{\sqrt{a}}+\dfrac{1}{\sqrt{b}}\right)$?
4. (14QIII5) Find the minimum of $2a^8+2b^6+a^4-b^3-2a^2-2$, where a and b are real numbers.
5. (10NA5) Let $x, y \in \mathbb{R}^+$ such that $x+2y=8$. Determine the minimum value of $x+y+\dfrac{3}{x}+\dfrac{9}{2y}$.
6. (15AI19) Find the maximum of $(1-x)(2-y)(3-z)\left(x+\dfrac{y}{2}+\dfrac{z}{3}\right)$ where $x<1, y<2, z<3$, and $x+\dfrac{y}{2}+\dfrac{z}{3}>0$.
7. (16AI18) Given $f(1-x)+(1-x)f(x)=5$ for all real numbers x, find the maximum of $f(x)$.
8. (16ND2) Suppose $\dfrac{1}{2} \le x \le 2$ and $\dfrac{4}{3} \le y \le \dfrac{3}{2}$. Determine the minimum of $\dfrac{x^3y^3}{x^6+3x^4y^2+3x^3y^3+3x^2y^4+y^6}$.
9. (13N5) Let r and s be positive real numbers that satisfy $(r+s-rs)(r+s+rs)=rs$. Find the minimum of $r+s-rs$ and $r+s+rs$.

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Session 9: Geometry 2
compiled by Carl Joshua Quines
October 19, 2016

Ad hoc

1. (16QIII1) In the right triangle ABC, where $\angle B = 90^\circ, BC : AB = 1 : 2$, construct the median BD and let point E be on BD such that $CE \perp BD$. Determine $BE : ED$.

2. (14NA9) A circle with diameter 2 is tangent to both diagonals of a square with side length of 2. The circle intersects the square at points P and Q. Find the length of segment PQ.

3. (9N5) Segments AC and BD intersect at point P such that $PA = PD$ and $PB = PC$. Let E be the foot of the perpendicular from P to the line CD. Prove that the line PE and the perpendicular bisectors of PA and PB are concurrent.

4. (10N2) On a cyclic quadrilateral $ABCD$, there is a point P on side AD such that the triangle CDP and the quadrilateral $ABCP$ have equal perimeters and equal areas. Prove that two sides of $ABCD$ have equal lengths.

5. (8N3) Let P be a point outside a circle, and let the two tangent lines through P touch the circle at A and B. Let C be a point on the minor arc AB, and let ray PC intersect the circle again at another point D. Let L be the line that passes through B parallel to PA, and let let L intersect rays AC and AD at points E and F, respectively. Prove that B is the midpoint of EF.

Triangles

1. (15AI5) Triangle ABC has a right angle at B, with $AB = 3$ and $BC = 4$. If D and E are points on AC and BC, respectively, such that $CD = DE = \frac{5}{3}$, find the perimeter of quadrilateral $ABED$.

2. (16AI11) Circle O is inscribed in the right triangle ACE with $\angle ACE = 90^\circ$, touching sides AC, CE and AE at points B, D and F, respectively. The length of AB is twice the length of BC. Find the length of CE if the perimeter of ACE is 36 units.

3. (8AII2) Let ABC be an acute-angled triangle. Let D and E be points on BC and AC such that $AD \perp BC$ and $BE \perp AC$. Let P be the point where ray AD meets the semicircle constructed outwardly on BC, and Q be the point where ray BE meets the semicircle constructed outwardly on AC. Prove that $PC = QC$.

4. (9AII3) The bisector of $\angle BAC$ intersects the circumcircle of triangle ABC again at D. Let AD and BC intersect at E, and F be the midpoint of BC. If $AB^2 + AC^2 = 2AD^2$, show that $EF = DF$.

5. (11N2) In triangle ABC, let X and Y be the midpoints of AB and AC, respectively. On segment BC, there is a point D, different from its midpoint, such that $\angle XDY = \angle BAC$. Prove that AD is perpendicular to BC.

Coordinate geometry

1. (16QII3) Let S be the set of all points A on the circle $x^2 + (y-2)^2 = 1$ so that the tangent line at A has a non-negative y-intercept; then S is the union of one or more circular arcs. Find the total length of S.

2. (15AI7) Find the area of the triangle having vertices $A(10, -9), B(19, 3)$, and $C(25, -21)$.

3. (16AII3) Point P on side BC of triangle ABC satisfies $BP : PC = 2 : 1$. Prove that the line AP bisects the median of triangle ABC drawn from vertex C.

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Session 10: Metasolving

compiled by Carl Joshua Quines

October 21, 2016

Best practice

1. Reread the question.
2. Work cleanly.
3. Be aware of your time.
4. Check your work.
5. Learn how to guess.

1 Reread the question

How to read

- The most common source of mistakes is misreading. ("I thought it was 2016, not 3^{2016}, I didn't see that it's supposed to be nonzero, I picked the wrong choice.") Thus, the most efficient way to reduce mistakes is to start by reducing the mistakes from misreading the question.
- When you first read the question, remember to take note of important details. One of the easiest ways to slip is to mistake an integer for a positive integer, or a complex number for a real number, to ignore bounds for a variable or ignore to read a condition. If you are not making progress on the problem, read it again – perhaps you missed an important detail somewhere.
- When answering, make sure to remember all the details given. More often than not, a detail given is important in answering the problem. There are very few exceptions to this, when more information is given than what is needed. In fact, if you didn't use all the information, you should be suspicious and check your work – perhaps there's a mistake.
- Once you have an answer, reread the question, even if you understand it completely. Is your answer in the proper format? Is it in the correct units? Does it match up with the choices, if there are any?

2 Work cleanly

How to write

- The next most common error of mistakes is misreading your handwriting. If you thought misreading the question was bad enough, misreading your handwriting is even worse. ("I thought I wrote a 2 instead of a 5, wait, where was my work for problem 9, is this a sine or a cosine?")
- Write neatly. This is the correct solution to half of your problems in handwriting. It doesn't have to be very pretty, it just needs to be legible enough for you to understand what you wrote.
- The reason for legible handwriting is simple enough: you need to understand your own handwriting. When checking your work, you will refer to what you wrote earlier. When choosing the answer, you will refer to your solution. Thus legible handwriting is needed.
- Make sure your handwriting is unambiguous. For example, you can write the lowercase L as ℓ instead of l, which is easier to understand if you're writing quickly. Adding the serifs in the digit 1 also helps. Finally, make sure your x and y look different enough to be differentiable when written quickly.

Organize

- The correct solution to the *other* half of your handwriting problems is organizing your scratch work. Yes, your *scratch work* should be organized as well.
- Draw boxes on your paper alloted to each problem. Problem 1 for the first box, problem 2 for the second, and so on. Or at the very least, indicate with an encircled number which problem is which, and try to keep your scratch work legible. This will help a lot when you're checking your answers later.
- Do not, *do not* try to cram all of your scratch work in a few pages of paper. You can always ask for more scratch paper, so if your paper is too crowded, don't add even more writing.

3 Be aware of time

Time management

- The principle of time management is trade-off. How much time should you spend solving and how much time should you spend checking your work? Should you use your time to solve this problem or that problem? It is important to make decisions like this quickly and accurately.
- Make sure you always have a rough idea of how much time is left. Wear a watch, and take note of what time the exam starts and ends. Usually the proctor will write overhead the time left, but it's always nicer to have a reference of how much time is left exactly.

4 Check your work

Checking

- Checking is important. Correcting a mistake is faster and easier than solving a new problem.
- Your checking method must be fast. Checking your work on one problem is always sacrificing time for the next problem, thus you must have a fast way to check.
- Mark problems that you are unsure about. That way, you know to spend more time checking problems you are unsure rather than sure about. Also, if you do not have enough time to check all your problems, then check only those you are not sure about.
- Do not repeat your solutions when you check. Find a different way, use a different method to check. If you use the same method, the tendency is for your brain to follow the same path, and thus repeat the same reasoning, and thus repeat the mistake. If you can't find a different method, find a different order.

Meta-checking

- If you finish the exam early, check your answers.
- If you have a few minutes left and the next problems are too difficult for you, check your answers.
- If you have a few minutes left, and you are a super-accurate person who never make mistakes, it is better to continue solving than to waste time checking your answers.
- If you rarely catch your own mistakes when checking your answers, it doesn't make sense to check.
- If the contest does not penalize wrong answers, it is best to guess strategically.
- Remember that correcting an error takes less time than solving a new question. Double-checking your work is usually a more efficient use of your time than proceeding to the next question. Besides, problems that were solved are easier than problems yet to be solved, so finding a mistake is faster than solving a new problem.
- Try to balance your time. If you are very error-prone, it is better to include more checking time at the end of the exam than to continue solving. On the other hand, if you are accurate, it is better to continue answering, or increase your speed.

Checking methods

- Plugging in the answer is the clearest, simplest method. Use this method whenever possible. In problems such as finding the value of x, sometimes plugging in is needed to determine if there are extraneous solutions.
- Plug in an intermediate result. Sometimes the answer asks for something that you can't plug in – for example, looking for $(2x+1)^2$ when you solve for x in an equation. In this case, you can't plug in the answer, but you can plug in x.
- Use a different method. For example, complementary counting. If we want to count how many two-digit positive integers that have at least one 7 as a digit, we can either do it directly (one digit containing 7, both digits containing 7...) or we can do it indirectly (number of two-digit numbers minus the two-digit numbers without 7).
- Use an example. Consider this problem: two non-zero real numbers, a and b, satisfy $ab = a - b$. Find a possible value of $a/b + b/a - ab$. If you solved this problem using algebra, it may help to use an example, such as $a = 1, b = 1/2$.

5 Learn how to guess

Guessing

- Eliminating one out of four choices improves your chances from 25% to 33%. Eliminating two out of four choices improves it to 50%, which is double the initial chances.
- Guess with caution in an exam that penalizes guessing. For example, in the American Mathematics Competitions, 1.5 points are awarded if a problem is left blank but 0 points are awarded if a problem is incorrect. In this case, if you cannot improve your guess, it is better not to guess.
- Always try to solve the problem without choices first. Sometimes the choices distract from solving the problem. Consider this problem: the sum of four two-digit numbers is 221, none of the eight digits are 0 and no two of them are the same. Which of the following is not included among the eight digits?

  (A) 2 (B) 4 (C) 6 (D) 8

  If you consider the choices, you might be tempted to do it by trial-and-error. But if you consider the problem alone, the idea of divisibility by 9 is immediate.
- Get into the mind of the problem-setter. The problem-setter wants to place choices that can confuse and mislead, and answers that are likely to be the result of a miscalculation or error in reasoning. Consider this problem: a digit watch displays hours and minutes with AM and PM. What is the largest possible sum of the digits in the display?

  (A) 17 (B) 19 (C) 21 (D) 23

  The common mistake is to assume $12:59$ produces the largest sum, leading to the answer 17. In a problem asking for maximums, the problem-setter will assume that some students will choose a smaller number over a larger one. That means the designers would include this mistake in the choices. Thus it is unlikely that the smallest answer will be correct – so you should think twice before answering 17. In this case, the correct answer is the last one, 23, corresponding to $9:59$.

Meta-guessing

- Consider the following choices:

  (A) $(-2,1)$ (B) $(-1,2)$ (C) $(2,-1)$ (D) $(1,-2)$ (E) $(4,4)$

  The pair $(4,4)$ is a clear outlier. Outliers are most likely not correct choices. If $(4,4)$ were the correct answer, then instead of solving the problem, we can merely use intermediate arguments. For example, if you can argue that both numbers must be positive, or that both numbers must be even, you can get the correct answer without solving the problem – that would be poor problem design. Thus $(4,4)$ should be eliminated during guessing.

- Consider the following choices:

  (A) $\frac{4}{9}$ (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ (D) $\frac{5}{6}$ (E) $\frac{9}{4}$

  Test designers want to create choices that appear correct. They try to anticipate possible mistakes. In this set of choices, the mistake they are hoping is confusing a number for its inverse. In this case, we can eliminate $5/6$, otherwise $6/5$ would be included. A similar example:

  (A) -2 (B) $-\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) 2

  The designers probably hope that students will confuse numbers with their inverses and negations. Thus we exclude the choice $1/3$.

- Sometimes the outlier might hint at the correct answer. Consider these choices:

  (A) 2 (B) $\frac{1}{2}\pi$ (C) π (D) 2π (E) 4π

  The outlier here is 2, all other choices have π. The problem designers were probably considering that the student might forget to multiply by π. Hence the likely correct answer is 2π, the answer we get by multiplying 2 and π.

Educated guessing

- The most common method in solving without actually solving is by example. Consider this problem: the difference between the squares of two odd numbers is always divisible by:

  (A) 3 (B) 5 (C) 6 (D) 7 (E) 8

  Choose 1 and 3 as our odd numbers, we see the difference of the squares is 8, thus the answer must be the last choice.

- Sometimes the choices help in solving the problem. Consider this: two non-zero real numbers, a and b, satisfy $ab = a - b$. Find a possible value of $a/b + b/a - ab$.

  (A) -2 (B) $-\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) 2

  Seeing the choices, we see that the answer is a number. We plug in $a = 1$ to see that $b = 1/2$ so the answer is 2. We are sure this is the only answer because only one answer can be correct.

- Here's another problem: let a, b, c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. Find $a^2 - b^2 + c^2$.

  (A) 0 (B) 1 (C) 4 (D) 7 (E) 8

  Again, seeing the choices, the answer is a number. Thus we can simply let c be anything, for example, we can let $c = 0$ to reduce it to a linear equation, and then we can compute the answer.

- Instead of solving the problem, sometimes we can plug in the solutions one-by-one and see which one works. Consider this problem: in triangle ABC, BD is the angle bisector of $\angle ABC$, and $AB = BD$. Moreover, E is a point on AB such that $AE = AD$. If $\angle ACB = 36^\circ$, find $\angle BDE$.

  (A) 24° (B) 18° (C) 15° (D) 12°

  The easiest method is to simply plug in each of the answers as $\angle BDE$ and see which one does not result in a contradiction, which gives us the answer.

Choice elimination

- Parity is one way to eliminate wrong answers. Consider the difference between the sum of the first 2003 even counting numbers and the sum of the first 2003 odd counting numbers.

  (A) 0 (B) 1 (C) 2 (D) 2003 (E) 4006

  Without computation we see the answer must be odd. Thus we can exclude three of the choices already. Parity is also an easy way to check your answers, as outlined below.

- Partial knowledge is helpful in solving. Consider solving for the number of non-negative integral solutions to $a+b+c=6$.

  (A) 22 (B) 25 (C) 27 (D) 28 (E) 29

  Even if you do not remember the exact formula, you may remember that the answer is a binomial coefficient that involves the number 6. The only choice that appears in the first 10 rows of Pascal's triangle is 28.

- Estimation is an important tool. Sometimes we do not need an exact answer to eliminate a wide range of choices. Consider this: let n be a 5-digit number, and let q and r be the quotient and remainder, respectively, when n is divided by 100. For how many values of n is $q+r$ divisible by 11?

  (A) 8180 (B) 8181 (C) 8182 (D) 9000 (E) 9090

  By the meta-guessing rule of removing outliers, we want to remove the last two choices: they are the only two that at least 9000. We confirm this with a quick estimation: there are 90000 5-digit numbers and about $90000/11 = 8182$ of them are divisible by 11, so we estimate near the first three choices.

- Sometimes you can eliminate all the choices as to produce the correct answer, without solving the problem. Consider solving for the non-zero value of x that satisfies $(7x)^{14} = (14x)^7$.

  (A) $\frac{1}{7}$ (B) $\frac{2}{7}$ (C) 1 (D) 7 (E) 14

  This can be solved very quickly: 1/7 does not work because the left-hand side will become 1 while the right-hand side will not, 1 and 7 cannot work because left-hand side will be odd and the right-hand side will be even, and finally 14 will make the right side too big. Thus the answer is 2/7, no solving needed.

- A final example: suppose we want to know the product of all odd positive integers less than 10000.

  (A) $\frac{10000!}{5000!^2}$ (B) $\frac{10000!}{2^{5000}}$ (C) $\frac{9999!}{2^{5000}}$ (D) $\frac{10000!}{2^{5000}5000!}$ (E) $\frac{5000!}{2^{5000}}$

  By the outlier-removal rule, we would remove the middle choices and the last choice, as we suspect that the numerator should be 10000!. We confirm this: take a prime number and see what power it belongs to the answer. Let's see we have a prime number p that is slightly below 5000. Then p should appear in the answer exactly once ($2p$ would be even, $3p$ would be too big).

  Now we look at the choices. Prime p appears in 5000! once as the factor p, and it appears in 10000! twice as p and $2p$. It appears in choice (A) zero times, in choice (B) and (C) twice, so the answers are either (D) or (E). But we can eliminate choice (E) as it does not contain any odd primes between 5000 and 10000, thus the answer is (D), confirming our meta-guess.

Quick reminders

- The PMO Qualifying Round is tomorrow, October 22. Please come on time. Latecomers will be left behind, for real. Okay, probably not. But don't be late anyway.
- Don't forget to bring a pen and a pencil. Last year's test required a pencil, best to be safe. And please make our school look decent, and remember to bring sharpeners and erasers, rulers and compasses, or whatever paraphernalia you need – it's degrading to borrow from one's seatmate.
- Eat well and gets lots of sleep. Don't get sick (which is what happened to me the first time I joined the PMO). Worry about schoolwork *after* the PMO – after all, it's the sembreak.

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